General Relativity: Tutorial/ Discussion

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I. Introduction to Hawking radiation

A. Introduction

- In this tutorial we shall clarify many aspects of the Hawking radiation effect by considering mainly flat space physics. This way of looking at the problem lies in the heart of the equivalence principle and proves to be very useful in the understanding of black hole radiation. A uniformly accelerated observer in Minkowski space cannot distinguish the force producing his non-inertial trajectory from a fictitious gravitational field.

- Moreover, this observer experiences the presence of a horizon in the same way as an exterior observer in the Schwarzschild geometry. Therefore a natural question arises: does the fictitious gravitational field, and the associated (observer-dependent) horizon, produce particles in analogy with the Hawking effect? The answer is positive and the associated phenomena [Fulling (1973); Davies (1975); Unruh (1976)] is usually called the Unruh effect.

- Today we shall try to explain the above analogy in the context of the so called near horizon approximation. This is a way to understand better, and from new viewpoints, the Hawking effect.

- We first describe the more standard (static) picture of Rindler space arising from the near horizon approximation of the Schwarzschild metric and the emergence of the infinite dimensional conformal symmetry of matter fields in the vicinity of the horizon.

- Then we analyse the physical consequences in dealing with a dynamical situation mimicking the gravitational collapse. We shall show in detail how the Hawking radiation and its properties can be understood in terms of the transformation law properties of the correlation functions of matter under spacetime conformal transformations.

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II. Near horizon geometry of the Schwarzschild BH

The introduction of the Rindler geometry can be motivated, from the physical point of view, from the important observation that the $(t-r)$-part of the Schwarzschild metric. Please note metric signature as of now is considered to be: $(-,+,+,+)$.

$$ds^2_{(4)} = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1-\frac{2M}{r})} + r^2 d\Omega^2 \tag{1}$$

is well approximated, around the horizon, by flat space. Introducing the coordinate $0 < x \ll 2M$ defined by,

$$r = 2M + \frac{x^2}{8M}, \tag{2}$$

and expanding the Schwarzschild metric around $2M$

$$1-\frac{2M}{r} = \frac{x^2}{16M^2} + O\left(\left(\frac{x}{M}\right)^4\right) \tag{3}$$

one gets,

$$ds^2_{(4)} \sim -(\kappa x)^2 dt^2 + dx^2 + (2M)^2 d\Omega^2, \tag{4}$$

- Derive Eq.(4)

where the constant κ coincides with the surface gravity of the Schwarzschild metric ($\kappa = 1/4M$). The first two terms of the above metric define a two-dimensional flat spacetime known as Rindler space [Rindler (1966); Rindler (2001)]. The last term describes a two-sphere with radius $2M$. It is easy to see that the apparent singularity at $x = 0$ can be eliminated introducing the null Minkowskian coordinates (U, V)

$$U = -x\, e^{-\kappa t} \qquad\qquad V = +x\, e^{\kappa t}, \tag{5}$$

with respect to which the (two-dimensional) Rindler metric becomes:

$$ds^2 = -dU dV. \tag{6}$$

It is convenient to introduce now a new coordinate ξ defined by the relation $x = \kappa^{-1} e^{\kappa\xi}$. The metric in the (t, ξ) coordinates reads:

$$ds^2 = e^{2\kappa\xi}(-dt^2 + d\xi^2). \tag{7}$$

- Derive Eq.(7)
- Recall: $ds^2_{\mathrm{R}} = e^{2a\xi}\left(d\tau^2 - d\xi^2\right)$.
- Note that the acceleration $a \equiv \kappa$, where κ is known as the surface gravity of the BH, which in turn depends on the mass of the BH.
- Thus we obtain the Rindler spacetime metric in $(1+1)$ dimensions, which emerges from the near horizon limit of the Schwarzschild BH. Therefore we have a clear and motivation to study the particle production from such a curved spacetime as the result from Rindler spacetime has already been established.

III. Near horizon geometry of the Reissner-Nordstr$\ddot{o}$m (RN) BH

We shall now jump to another physically interesting situation arising in the context of the near-horizon approximation. As we know the Schwarzschild geometry is not the only known black hole solution. By considering the Einstein–Maxwell theory we have the more general RN solution described by the line element as below:

$$ds^2_{(4)} = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{8}$$

where Q is the electric (or magnetic) charge. The main difference with respect to the Sch. spacetime is that now we have two horizons $r_\pm = M \pm \sqrt{M^2 - Q^2}$ for $M > |Q|$ (non-extremal case) and in the limiting situation $M = |Q|$ (extreme case) one degenerate horizon ($r_+ = r_-$). Let us now study the near-horizon geometry in close analogy with Schwarzschild. The analysis is slightly more involved due the presence of the two horizons.

A. Anti-de Sitter space as a near horizon geometry

A natural way to perform the near horizon limit is to expand around the outer horizon:

$$r = r_+ + x$$

. for $x \ll r_+$. We find then that, for non-extremal black holes ($r_+ \neq r_-$) the spacetime metric turns out to be:

$$ds^2_{(4)} \sim -\frac{x}{r_+}\left(1 - \frac{r_-}{r_+}\right)dt^2 + \frac{r_+}{x\left(1 - \frac{r_-}{r_+}\right)}dx^2 + r_+^2 d\Omega^2. \tag{9}$$

- Derive the metric in Eq. (9)

A redefinition of the x coordinate leads us to:

$$x \to \frac{(r_+ - r_-)}{4r_+^2}x^2 \tag{10}$$

and puts the metric in the Rindler-like form

$$ds^2_{(4)} = -(\kappa_+ x)^2 dt^2 + dx^2 + r_+^2 d\Omega^2, \tag{11}$$

where $\kappa_+ = \frac{(r_+ - r_-)}{2r_+^2}$ is the surface gravity at the horizon.

1. *Extremal black holes*

For extremal black holes ($r_+ = r_- \equiv r_0$) the approximated metric form can be achieved by replacing $r = r_0 + x$ for $x \ll r_0$ is:

$$ds^2_{(4)} \sim -\frac{x^2}{r_0^2}dt^2 + \frac{r_0^2}{x^2}dx^2 + r_0^2 d\Omega^2. \tag{12}$$

The above near horizon geometry for the extremal case is radically different from the Rindler geometry encountered for non-extremal black holes. In fact, the two-dimensional radial $(t - r)$ geometry in the Eq. (12) is non-flat and it has negative constant curvature. Thus we derive the Ricci scalar for the two dimensional part of the metric as written in the Eq. (12) as below:

$$R^{(2)} = -\frac{2}{r_0^2}. \tag{13}$$

- Derive the 2-dimensional Ricci scalar using the above metric

More precisely, it is just the portion of two-dimensional Anti-de Sitter space and now the whole $(3+1)$ dimensional spacetime looks like $\mathrm{AdS}_2 \times S^2$.

IV. Hawking radiation

BHs are massive objects which have such a strong gravitational field that even light cannot escape from them. According to the classical General Relativity, a black hole can only absorb matter and its size never decreases. In 1974 Hawking considered *quantum fields* in a *classical* black hole background and discovered that the black hole emits thermal particles and thus evaporates. This theoretical result came to a certain extent as a surprise. In fact, at that time one thought that particles can be produced only by a nonstatic gravitational field.

To understand why particle production is possible, we have to consider a virtual pair in the vicinity of the horizon. There are negative-energy states inside the horizon, and therefore one of the virtual particles (inside horizon) can have negative energy while the other one (outside horizon) has a positive energy. The first virtual particle can never escape from the black hole, but the second one can move away from the horizon to infinity thus becoming a real particle. As a result, the black hole can emit radiation and its mass decreases.

A. Study of Sch. BH spacetime

In this section we derive the Hawking result for a massless scalar field in two-dimensional spacetime. A 4-dimensional non-rotating black hole without electric charge is described by the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 \left(d\theta^2 + d\varphi^2 \sin^2\theta\right), \tag{14}$$

where M is the mass of a black hole (from now on we use the natural units where $G = \hbar = c = 1$). To simplify the calculations, we first consider a two-dimensional black hole, assuming that its metric is the same as the time-radial part of the Schwarzschild metric:

$$ds^2 = g_{ab}\, dx^a dx^b = \left(1 - \frac{r_g}{r}\right) dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}}, \tag{15}$$

where $r_g = 2M$. It is convenient to introduce the "tortoise coordinate" $r^*(r)$ according to

$$dr^* = \frac{dr}{1 - \frac{r_g}{r}}. \tag{16}$$

so that

$$r^*(r) = r - r_g + r_g \ln\left(\frac{r}{r_g} - 1\right). \tag{17}$$

Metric of Eq. (15) then becomes:

$$ds^2 = \left(1 - \frac{r_g}{r(r^*)}\right) \left[dt^2 - dr^{*2}\right], \tag{18}$$

where r must be expressed through r^* using Eq. (16). The tortoise coordinate r^* is defined only for $r > r_g$ and varies in the range $-\infty < r^* < +\infty$. As r approaches r_g the coordinate r^* goes to $-\infty$ and far away from the BH $r^* \to r$ as $r \to \infty$. Introducing the tortoise lightcone coordinates

$$\tilde{u} \equiv t - r^*, \quad \tilde{v} \equiv t + r^*, \tag{19}$$

we can rewrite metric in Eq. (15) in the form as follows,

$$ds^2 = \left(1 - \frac{r_g}{r(\tilde{u}, \tilde{v})}\right) d\tilde{u}\, d\tilde{v}. \tag{20}$$

B. Kruskal–Szekeres (KS) coordinates

The Schwarzschild coordinates are singular on the horizon at $r = r_g$. The tortoise lightcone coordinates $\tilde{u}, \tilde{v}$ are also singular and they cover only the "exterior" of the black hole, $r > r_g$. To describe the entire spacetime, we need another coordinate system. It follows from the Eqs. (17) and (19) that

$$1 - \frac{r_g}{r} = \frac{r_g}{r} \exp\left(1 - \frac{r}{r_g}\right) \exp\left(\frac{\tilde{v} - \tilde{u}}{2r_g}\right), \tag{21}$$

and hence metric (20) can be rewritten as,

$$ds^2 = \frac{r_g}{r} \exp\left(1 - \frac{r}{r_g}\right) e^{-\frac{\tilde{u}}{2r_g}}\, e^{\frac{\tilde{v}}{2r_g}}\, d\tilde{u}\, d\tilde{v}. \tag{22}$$

In the KS lightcone coordinates, defined as

$$u = -2r_g \exp\left(-\frac{\tilde{u}}{2r_g}\right), \quad v = 2r_g \exp\left(\frac{\tilde{v}}{2r_g}\right), \tag{23}$$

this metric takes the form

$$ds^2 = \frac{r_g}{r(u,v)} \exp\left(1 - \frac{r(u,v)}{r_g}\right) du\, dv \ , \tag{24}$$

and becomes regular at $r = r_g$. Thus the singularity that occurs in the Sch. metric as $r \to r_g$ is merely a coordinate singularity, which can be removed by a coordinate transformation. A freely falling observer will see nothing peculiar while crossing the horizon. As defined in the Eq. (23), the KS coordinates vary in the intervals $-\infty < u < 0$ and $0 < v < +\infty$, covering the "exterior" of the black hole at $r > r_g$. However, they can be analytically extended to $u > 0$ and $v < 0$, where metric (24) is still well-defined. The KS coordinates span the ranges $-\infty < u < \infty$ and $-\infty < v < \infty$, and thus cover the whole Sch. spacetime.

C. Field quantization and Hawking radiation: Choice of quantum state

To study the hawking radiation one can consider that the behaviour of a quantum field in the background spacetime of a collapsing body. However, the features of the Hawking effect turned out to be independent of the details of collapse, which suggests that the effect si more a consequence of the causal and topological structure of spacetime than the specific geometry. This indeed turns out to be the case.

We consider a scalar field with the action

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\sqrt{-g}d^2x \tag{25}$$

in a two-dimensional spacetime. Since this action is conformally invariant, the solution of the scalar field equation can be written either in terms of the lightcone tortoise coordinates (19) as,

$$\phi = \tilde{A}(\tilde{u}) + \tilde{B}(\tilde{v}), \tag{26}$$

or in the lightcone KS coordinates as

$$\phi = A(u) + B(v), \tag{27}$$

where $A, \tilde{A}$ etc. are arbitrary smooth functions. The situation is similar to the case of the scalar field in the Rindler spacetime. In particular,

$$\phi \propto e^{-i\omega\tilde{u}} = e^{-i\omega(t-r^*)} \tag{28}$$

describes a right-moving positive-frequency mode with respect to time t, which propagates away from the black hole. The proper time of an observer at rest located far away from the black hole coincides with t since

$$ds^2 \to d\tilde{u}d\tilde{v} = dt^2 - dr^{*2}$$

as $r \to \infty$ (see (15)). Therefore this observer associates "particles" with the positive frequency modes with respect to the time t. The expansion of the field operator,

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega\tilde{u}}\hat{b}_\Omega^- + e^{i\Omega\tilde{u}}\hat{b}_\Omega^+\right] + (\text{left-moving}), \tag{29}$$

determines the corresponding creation and annihilation operators $\hat{b}_\Omega^\pm$. As before, to simplify the formulae we do not write explicitly the contribution of the left-moving modes. The eigenstate $|0_B\rangle$ defined via

$$\hat{b}_\Omega^- \left|0_B\right\rangle = 0 \tag{30}$$

- is called the Boulware vacuum. It contains no particles from the point of view of the far away observer.
- The tortoise coordinates cover only the part of the Schwarzschild spacetime outside the black hole horizon. In this sense they are similar to the Rindler coordinates of an accelerated observer, and the Boulware vacuum is similar to the Rindler vacuum.

- Therefore the Boulware vacuum $|0_B\rangle$ is singular on the black hole horizon and hence it is physically unacceptable. In particular, the regularized energy density diverges on the horizon, and hence the backreaction of the quantum fluctuations makes the picture of a quantum field in the background of the almost unperturbed classical black hole inconsistent.

The Kruskal–Szekeres coordinates are nonsingular on the horizon and cover the whole Schwarzschild spacetime. In this sense they are similar to the inertial coordinates in Minkowski spacetime. In the vicinity of the horizon,

$$ds^2 \to dudv = dT^2 - dR^2, \qquad u = T - R, \qquad v = T + R \tag{31}$$

and particles registered by an observer crossing the horizon should be associated with positive-frequency modes with respect to the time T. Expanding the field operator in terms of the Kruskal-Szekeres lightcone coordinate,

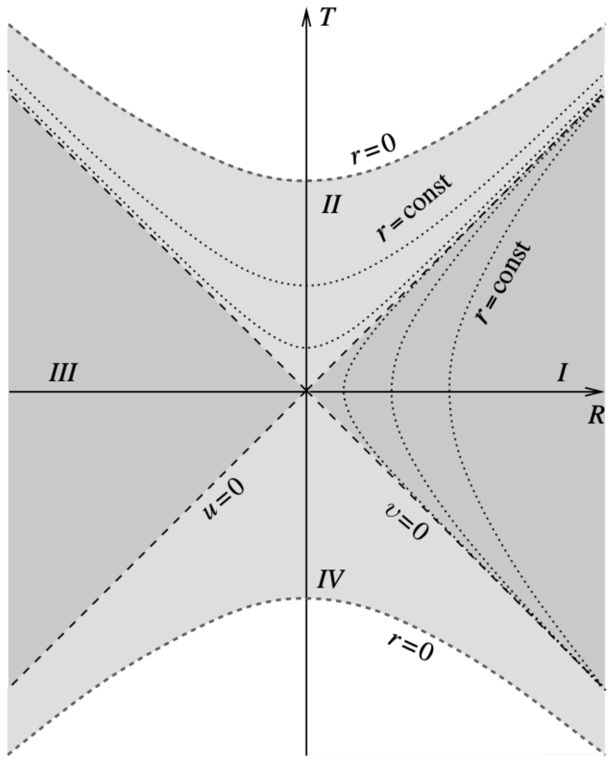

FIG. 1: A spacetime diagram in Kruskal–Szekeres coordinates (R, T). Shaded regions I–IV are the different asymptotic domains of the spacetime. Dashed lines represent the horizon $u = 0$ and $v = 0$. Dotted lines are surfaces of constant r. Thick dotted lines represent the singularity $r = 0$.

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u}\hat{a}_\omega^- + e^{i\omega u}\hat{a}_\omega^+\right] + (\text{left-moving}), \tag{32}$$

we define the creation and annihilation operators $\hat{a}_\omega^\pm$ that determine the Kruskal "vacuum" state,

$$\hat{a}_\omega^- |0_K\rangle = 0.$$

The state $|0_K\rangle$ is obviously regular on the horizon. Moreover, it leads to a finite energy density (after a subtraction of the zero-point energy), which for a large black hole is small everywhere except at the singularities. As a result, the backreaction of quantum fluctuations is negligible and they do not destroy the classical background. Hence the Kruskal state $|0_K\rangle$ is a natural candidate to be the true physical "vacuum" in the presence of the black hole.

From the point of view of a remote observer, the Kruskal vacuum $|0_K\rangle$ contains particles. To determine their number density, we can simply exploit the formal similarity between the formulae describing the accelerated observer and the black hole in two dimensions. Comparing the coordinate transformations for the Minkowski to Rindler spacetime we perceive that the Kruskal–Szekeres and tortoise lightcone coordinates are related exactly in the same way as the Minkowski and Rindler lightcone coordinates, if the acceleration a is replaced by the surface gravity $\kappa = (2r_g)^{-1}$. Then the mathematical similarity between the two cases becomes obvious:

Accelerated observer	Schwarzschild spacetime
Minkowski vacuum $\|0_M\rangle$	Kruskal vacuum $\|0_K\rangle$
Rindler vacuum $\|0_R\rangle$	Boulware vacuum $\|0_B\rangle$
Acceleration a	Surface gravity $\kappa = (2r_g)^{-1}$
$u = -a^{-1}\exp(-a\hat{u})$	$u = -\kappa^{-1}\exp(-\kappa\hat{u})$
$v = a^{-1}\exp(a\hat{v})$	$v = \kappa^{-1}\exp(\kappa\hat{v})$

Using this similarity and recalling the number operator calculation in Rindler spacetime, we find that the remote observer must see particles with the thermal spectrum,

$$\langle \hat{N}_\Omega \rangle \equiv \langle 0_K | \hat{b}^+_\Omega \hat{b}^-_\Omega | 0_K \rangle = \left[\exp\left(\frac{2\pi\Omega}{\kappa}\right) - 1\right]^{-1} \delta(0), \tag{33}$$

corresponding to the temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}.$$

Therefore, in the presence of quantum fields the picture of an eternal black hole is consistent only if the black hole is placed in a thermal reservoir with the temperature T_H. Because the black hole absorbs particles, it should also emit them to maintain the equilibrium.

V. Understanding hawking radiation for collapsing star

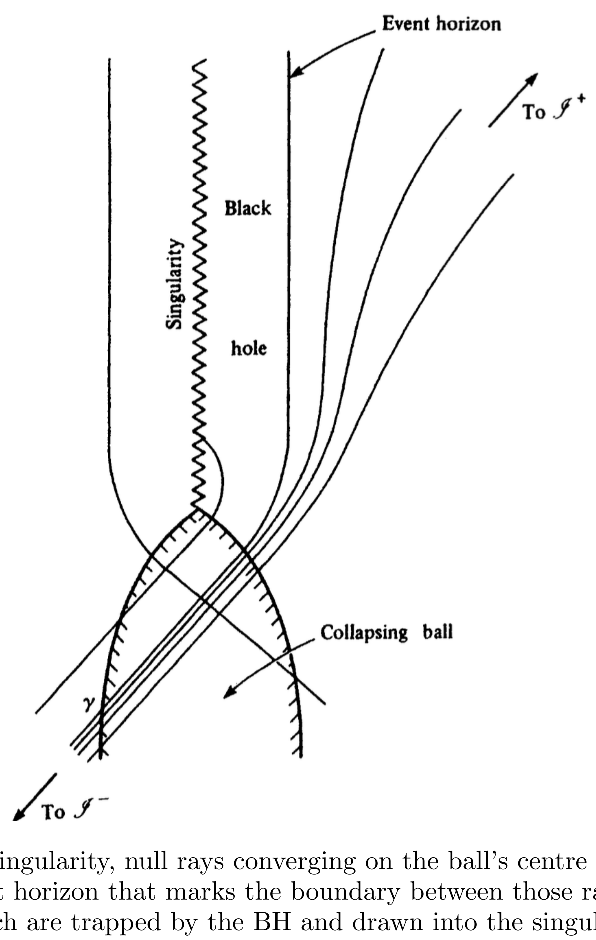

FIG. 2: As the ball collapses to a singularity, null rays converging on the ball's centre from $\mathcal{I}^-$ are distorted. One such ray, labelled, γ, forms the event horizon that marks the boundary between those rays that precede it and reach $\mathcal{I}^+$, and later rays which are trapped by the BH and drawn into the singularity: version 1

- Consider a spherically symmetric ball of matter surrounded by empty space. In the exterior region the unique spherically symmetric solution of Einstein's equation is the Schwarzschild spacetime. We do not worry here about the interior metric, as it will turn out to be unimportant.

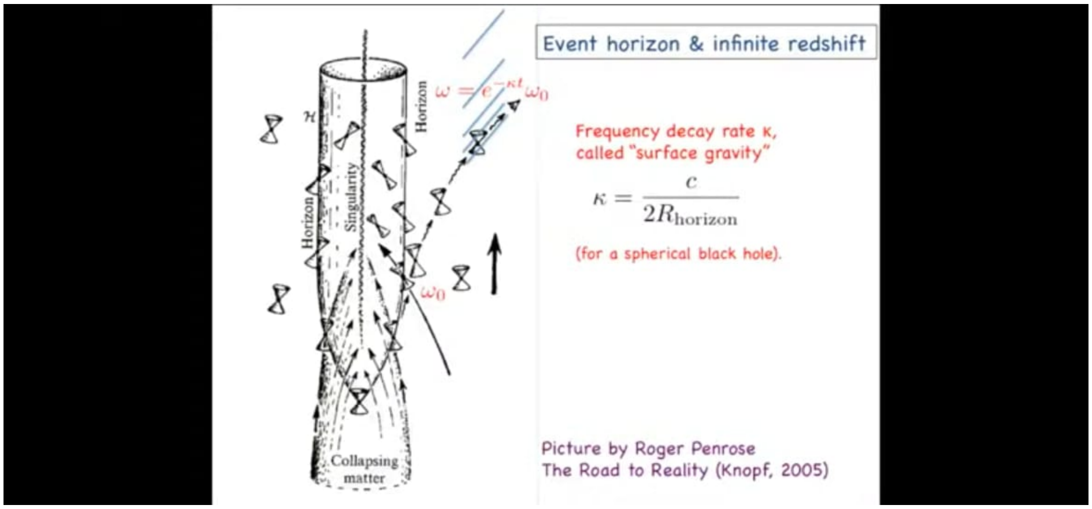

FIG. 3: Collapsing spherical body such as star and visualising Hawking radiation: version 2

- It is known (see, for example, Misner, Thorne & Wheeler 1973, chapter 31) that when sufficiently compact, the ball will implode catastrophically to form a Schwarzschild black hole.
- The exterior metric remains undisturbed by the collapse, but the modes of any quantum field propagating through the *interior* of the ball will be severely disrupted. Consequently, we expect particle production to take place.
- If it is assumed that in the remote past the ball is sufficiently distended that the spacetime is approximately flat, then one may construct the standard Minkowski space quantum vacuum state.
- After the collapse, the spacetime will have the Schwarzschild form and, in this out region, the vacuum will no longer correspond to the Minkowski space vacuum constructed in the in region. To calculate the particle production, one must compute the Bogolubov transformation between the in and out vacuum states in the usual way.
- To find the form of the field mode solution in the remote future, we first note that the incoming waves will converge on the centre of the ball, where they will pass on through to become outgoing spherical waves.
- As the incoming waves approach the surface of the ball they will suffer a blueshift, but when they re-emerge after passing through the ball, there will be a redshift. If the ball is static, these two effects exactly compensate and the waves would reach $\mathcal{I}^+$ with the same form as the initial one.
- However, if the ball is collapsing, during the time that the waves spend in transit through the ball, the ball will shrink somewhat, thereby raising its gravitational effect.
- The emerging waves will therefore suffer a redshift that is in excess of the blueshift acquired during the infall.
- The situation is depicted schematically in figs. (2) and (3), which shows the ball collapsing to form a singularity. Some ingoing null rays are shown passing through the centre of the collapsing ball and out the other side. There exists a latest advanced ray marked y, that just manages to penetrate the ball and reach $\mathcal{I}^+$ on the far side. This null ray forms the event horizon around the black hole.
- Later rays pass across the horizon and do not reach $\mathcal{I}^+$, but fall into the singularity instead. It is important to note that the direct interaction between the quantum field and the collapsing matter is being ignored. The presence of the matter in the model is used simply to produce an appropriate gravitational field.